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Ring waves on the surface of shear flows: a linear and nonlinear theory

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A theory is presented which describes the propagation of a ring wave on the surface of a flow which moves with some prescribed velocity profile. The problem is formulated in suitable far-field variables (which give the concentric KdV equation for a stationary flow), but allowance is made for the fact that the wavefront is no longer circular. The leading order of this small-amplitude long-wave theory reduces to a generalized Burns condition which is used to determine the shape of the wavefront. This condition is written as

$$(h^2+h'^2)\int_0^1 \mathrm{d}z/[F(z,\theta)]^2 = 1,$$

where $F(z,\theta) = -1+\{U(z)-c\}(h\cos\theta - h'\sin\theta)$, $U(z)$ is the velocity profile, c is a parameter and the local characteristic coordinate for the wave is $\xi = rh(\theta)-t$. (The Burns condition is interpreted in terms of the finite part of the integral in order to allow the possibility of a critical layer where $F(z_c,\theta) = 0, 0 < z_c < 1$.) The wavefront is represented by $r = \text{constant}/h(\theta)$. A model boundary-layer profile, which gives rise to a critical-layer solution, is chosen for $U(z)$. The role of this critical-layer solution, and the general question of upstream propagation, is then examined by constructing a wavefront which is continuous from the downstream to the upstream side. Solutions are presented which demonstrate that a critical layer never appears and so upstream propagation is necessary. These solutions (for various values of surface speed and boundary-layer thickness) are one branch of what we might term the singular solution of the differential equation for $h(\theta)$. The other branch corresponds to a solution which has a critical layer for all θ, which would seem to be unphysical since this solution is not an outward propagating ring wave.

At the next order we obtain the equation which describes the dominant contribution to the surface wave, in this approximation. The equation is a new form of Korteweg–de Vries equation; the novel feature is the dependence on the polar angle, θ. This equation is not analysed in any detail here, but the connection with plane waves over a shear flow, and with concentric waves in the absence of shear, is made.

1. Introduction

The various theories of surface waves are a fruitful area of investigation; for example, they may be developed with or without the inclusion of viscous effects or surface tension or variable depth. The development may attempt a description of nonlinear wave evolution, or incorporate any shear flow – perhaps even a model of turbulent flow – below the free surface. There are numerous choices and combinations. Of all these possibilities, many of them classical problems, one very familiar example is the description of circular-ring waves on the surface of a stationary fluid. This simple theory is at the heart of the Kelvin ship-wave pattern generated by a moving disturbance (see e.g. Stoker 1957 or Crapper 1984). Similarly, the analysis of the propagation of plane waves on the surface of a shear (i.e. rotational) flow – the flow direction being normal to the wave front – is also a well-understood problem. A discussion of this problem, in the limit of long waves, was given by Burns (1953). (Although Burns' name is usually associated with this problem, it was first examined by Thompson (1949).) In this theory the speed of the waves (c) is determined by an integral constraint of the form

$$\int_0^1 \frac{\mathrm{d}z}{[U(z)-c]^2} = 1,$$

when written in suitable non-dimensional variables; $U(z)$ is the given shear flow. (The use of the term 'shear flow' is not meant to suggest a velocity profile which is generated by a viscous shear stress but rather to indicate the type of rotational profile that could be chosen.) The conclusions which follow from this 'Burns condition' have been discussed by, for example, Benjamin (1957), Velthuizen & Wijngaarden (1969) and Yih (1972). It is well known that this integral equation for c certainly admits two solutions for c if $U(z)$ satisfies $U'(z) > 0$ and $U''(z) < 0$: one solution gives $c < U(0)$ and the other $c > U(1)$.

These two solutions of the Burns condition have generated some discussion, as will be seen in the papers cited above. The difficulty is that, if $U(1)$ is quite large as compared with $U(0)$ (which we may as well choose to be zero), then it is possible to have a disturbance which propagates upstream ($c < 0$) against a fast-flowing stream. Some arguments have been marshalled to suggest that this is unreasonable, but Yih (1972) confirms that $c < 0$ is a perfectly respectable solution. One possible alternative is to seek another solution which corresponds to the existence of a *critical layer* i.e. $0 < c < U(1)$; see e.g. Benney & Bergeron (1969). It is clear that the Burns condition as written cannot admit such a solution (since the integral will not be finite). However, a reformulation of the problem allowing for the presence of a critical layer *a priori* leads to the same integral constraint, but now interpreted as a finite part,

$$\fint_0^1 \frac{\mathrm{d}z}{[U(z)-c]^2} = 1; \qquad (1)$$

see Johnson (1986) for the details relevant to this problem. The solutions of this form of the Burns condition for various choices of the shear profile, $U(z)$, are given in Johnson (1990) (and some general conditions under which critical-layer solutions will occur are also discussed). The appearance of integrals evaluated in terms of their finite parts is a common feature of flows which contain a critical layer.

One standard procedure which is often adopted when analysing the relevance of various solutions of the Burns condition (or other similar propagation problems) is to examine an individual plane-wave propagating at a fixed speed (c). It is then usual to introduce suitable additional physical effects, such as those due to laminar viscosity or a Reynolds stress, in order to discuss the stability of the wave. The speed of propagation must then be treated as a complex number, $c = c_r + \mathrm{i}c_i$, and the possibility of $0 < c_r < U(1)$ as $c_i \to 0$ may be considered (see Lin 1955; Yih 1972). In this discussion, however, we shall adopt a different approach.

Let us suppose that the Burns condition, (1), admits three solutions, one of which is therefore associated with a critical layer so that $0 < c < U(1)$ (if, as before, we take $U(0) = 0$). This solution, like $c > U(1)$, then corresponds to downstream propagation; there is one solution describing upstream propagation ($c < 0$). We may then examine the strictly inviscid – but rotational – problem of how a ring wave propagates (outwards) in the presence of a given velocity profile beneath the surface. The furthermost downstream portion of the ring wave will propagate at the local speed $c > U(1)$, in the long-wave approximation. We suppose that the wavefront is continuous from the downstream to the upstream side, and we may then determine the form of the wavefront. This will, first, enable us to describe how the velocity profile (shear flow) affects the shape of the wavefront itself. Secondly, and of some considerable interest here, is the question of which solution (locally) of the Burns condition is relevant to the propagation near the rearmost portion of the wavefront. The ring wave may allow connection to either the critical-layer solution ($0 < c < U(1)$) or the solution $c < 0$. It is immediately clear that if the critical-layer solution is the favoured one then a critical layer will form on the bed of the flow at two symmetric points under the wavefront. This critical level will then rise off the bottom, attaining a maximum height at the rearmost point of the wavefront. On the other hand, if the connection is to the solution $c < 0$ then we might surmise that the flow remains non-critical everywhere. Thus our intention is to examine the problem of determining the shape of the wavefront (in the long-wave limit and in a suitable far field) for propagation over an arbitrary velocity profile, and the consequent existence or otherwise of a critical layer. It is not our intention to discuss the conventional hydrodynamic-stability aspects of this problem here.

The description of the problem that we have outlined above relates to the propagation of linear long waves in a far field. The far field corresponds to large radius, sufficiently large for the geometric decay of the ring wave to be relegated to the next order of approximation. (This is what happens in the derivation of the concentric KdV equation, mentioned below.) It would seem natural, therefore, to develop the analysis further and include nonlinear and other effects. The study of nonlinear long-wave motion is one which has been particularly successful in recent years, mainly because many of the resulting equations can be solved exactly. The most familiar equation of this type is the Korteweg–de Vries (KdV) equation which admits N-soliton solutions obtained from the Inverse Scattering Transform (IST) method. This classical KdV equation relates to one-dimensional wave propagation, but in cylindrical coordinates another form of KdV equation can be derived (the concentric KdV equation) and others are possible; see Johnson (1980) for a discussion of these various equations as they appear in water-wave theory. The problem of determining the form of KdV equation relevant to propagation over a shear flow is also not new; for the case of one-dimensional propagation, see Freeman & Johnson (1970). This work came out of an analysis to extend the earlier discussion of Benjamin (1962) describing the motion of a solitary wave on an arbitrary rotational flow. Here, by suitable choice of the scalings, we shall derive the KdV equation which describes the surface ring-wave (not necessarily circular) in the long-wave limit and for small amplitudes, in the presence of a shear flow below the surface. It is beyond the scope of this work to detail the properties of this new KdV equation, but we shall certainly see how it corresponds to the various special cases that are readily accessible.

Thus, in summary, we have three goals in mind. First, we hope to give a fairly detailed description of the shape of the wavefront as the ring wave propagates over a shear flow (in the long-wave approximation and at large radius). (The results of this calculation might then be relevant in obtaining the gross features of a velocity profile from observations of the details of the shape of the wavefront on the surface.) Of more significance, it is expected that the form of the wavefront will shed some light on the role of the various solutions of the Burns condition. We can hope to give, using our approach, another perspective on the question of upstream/downstream propagation; this is our second aim. Finally, we shall extend the small-amplitude long-wave linear theory to encompass a suitable far field where nonlinear, dispersive and geometric effects balance. This will lead to the derivation of the appropriate KdV equation valid for non-circular ring waves moving over an arbitrary shear flow.

2. Governing equations

The Euler equations, the equation of mass conservation, and the boundary conditions are written as

$$\left.\begin{aligned}&\frac{\mathrm{D}\hat{\boldsymbol{u}}_\perp}{\mathrm{D}t}+\frac{1}{\rho}\boldsymbol{\nabla}_\perp\hat{P}=0,\quad \frac{\mathrm{D}\hat{w}}{\mathrm{D}t}+\frac{1}{\rho}\hat{P}_z=-g,\quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}=0,\\ &\hat{w}=0\quad\text{on}\quad z=0,\quad \hat{w}=\frac{\mathrm{D}\hat{h}}{\mathrm{D}t},\quad \hat{P}=P_0\quad\text{on}\quad z=\hat{h}.\end{aligned}\right\}\tag{2}$$

Here, $\hat{\boldsymbol{u}}_\perp \equiv (\hat{u},\hat{v})$ represents the horizontal components only of the velocity vector, $\hat{\boldsymbol{u}} \equiv (\hat{u},\hat{v},\hat{w})$. Correspondingly $\boldsymbol{\nabla}_\perp$ represents the x and y components of the operator $\boldsymbol{\nabla}$; the third (vertical) component of $\boldsymbol{\nabla}$ is $\partial/\partial z$. We shall take ρ and g to be constants, with $\hat{P}=P_0=$ constant on the surface $z=\hat{h}(x,y,t)$. These equations are non-dimensionalized by writing

$$z\to h_0 z,\quad \hat{\boldsymbol{u}}_\perp\to(gh_0)^{\frac{1}{2}}\hat{\boldsymbol{u}}_\perp,\quad \hat{h}=h_0(1+\epsilon\hat{\eta}),$$

where h_0 is the undisturbed depth of water and ϵ is an amplitude parameter. In order to describe the long waves of interest here we also write

$$(x,y)\to(x,y)\,h_0/\epsilon^{\frac{1}{2}},\quad t\to(h_0/(gh_0)^{\frac{1}{2}})t/\epsilon^{\frac{1}{2}},\quad \hat{w}\to\epsilon^{\frac{1}{2}}(gh_0)^{\frac{1}{2}}\hat{w};$$

the pressure is expressed in terms of its deviation away from the hydrostatic pressure distribution,

$$\hat{P}=P_0+\rho g h_0(\hat{p}+1-z).$$

The governing equations, (2), now become

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}_\perp}{\mathrm{D}t}+\boldsymbol{\nabla}_\perp\hat{p}=0,\quad \epsilon\frac{\mathrm{D}\hat{w}}{\mathrm{D}t}+\hat{p}_z=0,\quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}=0,$$

$$\hat{w}=0\quad\text{on}\quad z=0,\quad \hat{w}=\epsilon\frac{\mathrm{D}\eta}{\mathrm{D}t},\quad \hat{p}=\epsilon\hat{\eta}\quad\text{on}\quad z=1+\epsilon\hat{\eta}.$$

It is convenient to introduce, at this stage, a polar-coordinate frame which is moving in the x-direction at a speed c,

$$x=ct+r\cos\theta,\quad y=r\sin\theta,\tag{3}$$

and c, it turns out, will play the role of a parameter in our analysis. Consistent with this choice of independent variables we write

$$\left.\begin{aligned}&\hat{u}=U(z)+\epsilon(u\cos\theta-v\sin\theta),\quad \hat{v}=\epsilon(u\sin\theta+v\cos\theta),\\ &\hat{w}=\epsilon w,\quad \hat{p}=\epsilon p,\end{aligned}\right\}\tag{4}$$

with

where the arbitrary shear flow in the x-direction is represented by $U(z)$. The dependence on ϵ in the transformations (4) is consistent with the description of small-amplitude motion as $\epsilon \to 0$.

The particular linear and nonlinear theory for a ring wave that we shall describe here arises when we make the choice in (r,θ,t) – space where $\xi = O(1)$, $R = O(1)$ with

$$\xi = rh(\theta)-t, \quad R = \epsilon rh(\theta). \tag{5}$$

The wavefront is therefore described by $rh(\theta)-t = \text{constant}$, and $h(\theta)$ is to be determined. With this selection of variables (which essentially corresponds to that used to derive the concentric KdV equation in Johnson (1980)) we find that the leading-order (linear) problem does not incorporate any geometric or dispersive effects. However, both these appear at the next order, in conjunction with nonlinear terms, thereby generating the appropriate KdV equation for this problem. The governing equations may now be written in the form

$$(\mathrm{D}_1+\epsilon\mathrm{D}_2)u+\epsilon(U-c)\frac{h}{R}v\sin\theta+U'w\cos\theta+\epsilon(\mathrm{D}_3+\epsilon\mathrm{D}_4)\,u-\epsilon^2\frac{hv^2}{R}+h(p_\xi+\epsilon p_R) = 0, \tag{6}$$

$$(\mathrm{D}_1+\epsilon\mathrm{D}_2)\,v-\epsilon(U-c)\frac{hu}{R}\sin\theta-U'w\sin\theta+\epsilon(\mathrm{D}_3+\epsilon\mathrm{D}_4)\,v+\epsilon^2\frac{huv}{R} +h'(p_\xi+\epsilon p_R)+\frac{\epsilon h}{R}p_\theta = 0, \tag{7}$$

$$p_z+\epsilon(\mathrm{D}_1+\epsilon\mathrm{D}_2)w+\epsilon^2(\mathrm{D}_3+\epsilon\mathrm{D}_4)w = 0, \tag{8}$$

$$hu_\xi+h'v_\xi+w_z+\epsilon\left(hu_R+\frac{hu}{R}+h'v_R+\frac{h}{R}v_\theta\right) = 0, \tag{9}$$

which are the r-, θ-, z-momentum equations and the equation of mass conservation, respectively. The subscripts denote partial derivatives, U' is $\mathrm{d}U/\mathrm{d}z$, h' is $\mathrm{d}h/\mathrm{d}\theta$ and the differential operators (D_n) are defined by

$$\mathrm{D}_1 \equiv [-1+\{U(z)-c\}\,(h\cos\theta-h'\sin\theta)]\frac{\partial}{\partial\xi},$$

$$\mathrm{D}_2 \equiv \{U(z)-c\}\left\{(h\cos\theta-h'\sin\theta)\frac{\partial}{\partial R}-\frac{h}{R}\sin\theta\frac{\partial}{\partial\theta}\right\},$$

$$\mathrm{D}_3 \equiv (hu+h'v)\frac{\partial}{\partial\xi}+w\frac{\partial}{\partial z}, \quad \mathrm{D}_4 \equiv (hu+h'v)\frac{\partial}{\partial R}+\frac{hv}{R}\frac{\partial}{\partial\theta}.$$

The boundary conditions are expressed as

$$w = 0 \quad \text{on} \quad z = 0, \quad p = \eta \quad \text{on} \quad z = 1+\epsilon\eta, \tag{10}$$

$$w = (\mathrm{D}_1+\epsilon\mathrm{D}_2)\eta+\epsilon(hu+h'v)\eta_\xi+\epsilon^2\mathrm{D}_4\,\eta \quad \text{on} \quad z = 1+\epsilon\eta, \tag{11}$$

where $\eta = \eta(\xi,R,\theta;\epsilon)$. The problem described by the equations (6)–(11) involves the single parameter ϵ; we shall construct the asymptotic solution to these equations in the limit as $\epsilon \to 0$. By virtue of the scalings that have been adopted this parameter, as it approaches zero, corresponds to the case of both small amplitude and long waves propagating in a far field (i.e. at large radius).

3. The linear problem

We seek a solution of these equations by expanding each dependent variable in the form
$$q \sim q_0 + \epsilon q_1 \quad \text{as} \quad \epsilon \to 0, \tag{12}$$
where $q \equiv u, v, w, p, \eta$. The leading-order equations from (6)–(11) will then describe an appropriate linear problem (in the long-wave limit); these equations are
$$-u_{0_\xi} + (U-c)(h\cos\theta - h'\sin\theta)\, u_{0_\xi} + U'w_0\cos\theta + hp_{0_\xi} = 0,$$
$$-v_{0_\xi} + (U-c)(h\cos\theta - h'\sin\theta) v_{0_\xi} - U'w_0\sin\theta + h'p_{0_\xi} = 0,$$
$$p_{0_\xi} = 0, \quad hu_{0_\xi} + h'v_{0_\xi} + w_{0_z} = 0,$$
with
$$w_0 = 0 \quad \text{on} \quad z = 0;$$
$$p_0 = \eta_0, \quad w_0 = -\eta_{0_\xi} + (U-c)(h\cos\theta - h'\sin\theta)\,\eta_{0_\xi} \quad \text{on} \quad z = 1.$$
We note that, by virtue of the definitions of ξ and R (see (5)), these equations are valid at a large radius ($O(\epsilon^{-\frac{3}{2}})$) but relatively close to the wavefront ($O(\epsilon^{-\frac{1}{2}})$). In consequence the geometric decay of the (classical) linear problem is not evident at the leading order as $\epsilon \to 0$. It is clear that $p_0 = \eta_0$ for $z \in [0, 1]$ and then, if we write
$$F(z, \theta) = -1 + \{U(z) - c\}(h\cos\theta - h'\sin\theta), \tag{13}$$
we see that the r- and θ-momentum equations, with the mass conservation equation, imply
$$-Fw_{0_z} + F_z w_0 + (h^2 + h'^2)\,\eta_{0_\xi} = 0.$$
(This has the same structure as in the classical Burns problem.) It now follows directly that
$$w_0 = \eta_{0_\xi}(h^2 + h'^2) F \int_0^z \frac{\mathrm{d}z}{F^2},$$
and then the surface boundary-condition on w_0 requires that
$$(h^2 + h'^2) \int_0^1 \frac{\mathrm{d}z}{[1 - \{U(z) - c\}(h\cos\theta - h'\sin\theta)]^2} = 1. \tag{14}$$
This integral constraint reduces to the well-known Burns integral if $h(\theta) = 1$ and $\theta = 0$; we shall refer to (14) as the generalized Burns condition. For our purposes we shall suppose that the velocity profile, $U(z)$, is prescribed and that the speed, c, of the frame of reference is to be chosen; equation (14) then becomes a first-order nonlinear ordinary differential equation for $h(\theta)$. At this order the leading-order representation of the surface wave, η_0, remains undetermined.

A simple comparison of our formulation with earlier work is afforded by the special case of a plane oblique wave moving over a shear flow. To see this we set $h = 1$ and take $\theta =$ constant and then, in particular, we may examine $z \to z_c^+$ where $z_c (0 < z_c < 1)$ is the critical level defined by
$$F(z_c, \theta) = 0.$$
It then follows from the leading-order equations that, for example,
$$v_0^+ \sim -\frac{\eta_0}{\epsilon^{\frac{1}{2}} Z} \frac{\sin\theta}{U_c' \cos^2\theta} \quad \text{as} \quad Z \to 0,$$

where $z - z_c = \epsilon^{\frac{1}{3}} Z$ and the subscript c denotes evaluation at the critical level. The corresponding result for u_0^+ shows that this component of the velocity is not singular as $Z \to 0$. Both these properties agree with those for the plane oblique wave given by Peregrine (1976). (Peregrine also mentions the special class of 'surface layer solutions' for which w_{0_z} is discontinuous at the critical level and $w_0 = 0$ below it. However, as Peregrine points out, these solutions are not relevant in the limit of long waves which is the assumption we make here; see Yih 1972.)

4. The generalized Burns condition

Earlier, reference was made to the form that the Burns condition takes if a critical layer is supposed to occur in the flow. That is, the integral is to be interpreted as defined by its finite part. In the work presented here the same situation obtains: if the analysis is carried through under the assumption that a critical layer occurs at $z = z_c, 0 < z_c < 1$, for a given θ, then the finite part of the integral is required. Thus the generalized Burns condition, (14), is rewritten as

$$(h^2 + h'^2) \int_0^1 \mathrm{d}z / [F(z, \theta)]^2 = 1, \tag{15}$$

where $F(z, \theta)$ is given by (13). We shall assume hereafter that a critical layer may be accommodated by choosing to use (15) as the generalized Burns condition.

Before we turn to a more detailed examination of this equation it is instructive to see how, for example, the critical-layer condition $F(z, \theta) = 0$ arises directly. We consider general wavefront propagation where the wavefront itself is given by

$$H(r, \theta, t) = rh(\theta) - t = \text{constant}.$$

The outward unit normal to this front is

$$\frac{\nabla H}{|\nabla H|} = \frac{h\boldsymbol{e}_r + h'\boldsymbol{e}_\theta}{(h^2 + h'^2)^{\frac{1}{2}}},$$

where $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ are unit vectors in the polar-coordinate directions. Also, the speed (outward) of the wave is

$$-\frac{H_t}{|\nabla H|} = \frac{1}{(h^2 + h'^2)^{\frac{1}{2}}}.$$

If this wave is in a frame of reference which is moving at a speed c in the x-direction, then the outward speed, in a fixed frame, normal to the front is

$$\frac{1}{(h^2 + h'^2)^{\frac{1}{2}}} + c \cos(\alpha + \theta),$$

where $\cos \alpha = h/(h^2 + h'^2)^{\frac{1}{2}}$, $\sin \alpha = h'/(h^2 + h'^2)^{\frac{1}{2}}$. Now a critical layer will occur where the propagation speed normal to the wavefront equals the speed of the shear flow in the same direction, i.e.

$$\frac{1}{(h^2 + h'^2)^{\frac{1}{2}}} + c \cos(\alpha + \theta) = U(z) \cos(\alpha + \theta),$$

or

$$\{U(z) - c\}(h \cos \theta - h' \sin \theta) - 1 = 0,$$

which is precisely $F(z, \theta) = 0$.

One further point: the classical Burns integral is an equation for c given $U(z)$, whereas in our presentation we argue that c may be arbitrarily assigned. The reason for this difference is that the classical Burns integral is valid for plane waves, i.e. the shape of the wavefront is fixed, but here we determine $h(\theta)$. In other words, the classical integral determines the speeds at which steady plane waves may propagate (in the long-wave, small-amplitude approximation). This well-known result is recovered from our work if we prescribe $h(\theta)$ (and θ) and then solve for c, e.g. set $h(\theta) \equiv 1$ and $\theta = 0$. In our formulation we are given $U(z)$ and then we choose the frame of reference (c); the existence of steady ring-wave solutions is then a question of the existence of $h(\theta)$. Of course, the form of $h(\theta)$ may be made simpler – or more complicated – by the choice of c, even though the final shape of the wavefront is the same (for all c) when expressed, for example, in fixed Cartesian coordinates. It turns out, not surprisingly, that the most convenient choice for c is $c = U(1)$, the surface speed of the flow.

We now turn to the generalized Burns condition, (15), and discuss its properties by choosing to work with a specific velocity profile. Our choice is one which admits a critical-layer solution and for which the integration is particularly simple. It is anticipated that other more realistic profiles – which also give a critical layer – will produce roughly similar results. Of course we make a choice here which leads to a straightforward calculation; other model profiles may not produce such transparent details. (For related profiles in connection with the classical Burns condition see Velthuizen & Wijngaarden (1969) and Johnson (1990).) The velocity profile we choose is

$$U(z) = \begin{cases} U_1, & d \leqslant z \leqslant 1, \\ U_1 z/d, & 0 \leqslant z < d, \end{cases} \tag{16}$$

which embodies all the salient features that we wish to exploit. From (15) and (16) we find directly that, for $0 \leqslant d \leqslant 1$,

$$(h^2 + h'^2)\,[1 - d + d/\{1 + U_1(h\cos\theta - h'\sin\theta)\}] = 1, \tag{17}$$

where the choice of frame, $c = U_1$, has been made. To set $c = U_1$ is, apart from being the choice which gives the simplest form of equation, also the most natural preference: the polar coordinate system is moving at the surface speed.

First, (17) is examined in the case $\theta = 0$ with $h'(0) = 0$ (a reasonable assumption for the solutions of interest) and then it is clear that the resulting cubic for $h(0)$ has only one positive root. This is the solution which is required near $\theta = 0$ in order to produce an outward propagating wave; in terms of the classical Burns condition it is equivalent to the solution $c > U_1$. Correspondingly, if $\theta = \pi$ and $h'(\pi) = 0$, then the roots for $h(\pi)$ are just the negative of those for $h(0)$ and hence there are now two possible solutions which are consistent with outward propagation. These two solutions are the counterparts, for the classical Burns condition, of $c < 0$ and $0 < c < U_1$. So, for example, if $d \to 0$ then

$$h(0) \sim 1 + \tfrac{1}{2}d\frac{U_1}{1+U_1}, \quad -1 + \tfrac{1}{2}d\frac{U_1}{1-U_1}, \quad -\frac{1}{U_1} - \frac{d}{U_1(1-U_1^2)}, \tag{18}$$

for $U_1 \neq 1$; if $U_1 = 1$ then

$$h(0) \sim 1 + \tfrac{1}{4}d, \quad -1 + (\tfrac{1}{2}d)^{\frac{1}{2}}, \quad -1 - (\tfrac{1}{2}d)^{\frac{1}{2}} \quad (d \to 0). \tag{19}$$

In fact, (17) can be solved for all θ quite easily in the case of a thin boundary layer; the relevant solution (i.e. $h(0) > 0$) is

$$h(\theta) \sim 1+\tfrac{1}{2}d\frac{U_1\cos\theta}{1+U_1\cos\theta} \quad (d\to 0), \tag{20}$$

if $0 \leqslant U_1 < 1$. On $\theta = \pi$ solution (20) gives

$$h(\pi) \sim 1-\tfrac{1}{2}d\frac{U_1}{1-U_1},$$

which is not the critical-layer solution near $\theta = \pi$. Thus the first wavefront profile that we have determined – albeit asymptotic – is continuous (and smooth) from $\theta = 0$ to $\theta = \pi$ (or to $\theta = -\pi$), and it is non-critical for all θ. However, there is little virtue in exploring this special limit ($d \to 0$) to any great extent since (17) can be solved completely for all $0 \leqslant d \leqslant 1$.

A closer inspection of (17) reveals that the general solution can be expressed as

$$h(\theta) = a\cos\theta + b(a)\sin\theta, \tag{21}$$

where

$$(a^2+b^2)\,[1-d+d/(1+aU_1)] = 1. \tag{22}$$

Of course, the generalized Burns condition, (15), itself can also be solved by (21) but with $b(a)$ now obtained from

$$(a^2+b^2)\int_0^1 \frac{\mathrm{d}z}{[a\{U(z)-c\}-1]^2} = 1.$$

If b is real then the two choices $b > 0, b < 0$ correspond to $\theta > 0, \theta < 0$; the sign of b is therefore essentially irrelevant. This might seem all quite satisfactory at first sight, but (21) and (22) are not the end of the story. It transpires that these equations do not admit a solution for which $h(\theta) > 0$ for all θ; if $h(\theta)$ were to change sign then the solution is partly an outward propagating wave and partly inward, which is unacceptable. Indeed, this would imply that $h(\theta) = 0$ for some θ and then the wavefront must extend to infinity ($r = (\text{const}+t)/h(\theta) \to \infty$). It is therefore inappropriate to suggest that the ring wave envisaged here (i.e. bounded and continuous) could be described by both inward and outward propagating regions. (We must remember that the true direction of propagation in the physical frame requires the velocity of the moving frame to be introduced.) Although there are various configurations of wavefronts for which $r \to \infty$, none of them are relevant to our problem; we require the fore and aft portions of the wavefront to be connected. It is the constraints imposed by requiring that we have a true ring wave, and then what this implies about local upstream/downstream propagation, that we are investigating. Of course, wavefronts which extend to infinity may be of interest in a different context.

To see how the relevant solution arises it is convenient to introduce a parameter. Let

$$h(p) = a(p)\cos p + b(p)\sin p,$$

and

$$h'(p) = -a(p)\sin p + b(p)\cos p,$$

then $\mathrm{d}h/\mathrm{d}p = h'(p)$ implies that

$$\frac{\mathrm{d}a}{\mathrm{d}p}\cos p + \frac{\mathrm{d}b}{\mathrm{d}p}\sin p = 0,$$

where a and b are related by (22). Thus, either $a'(p) = b'(p) = 0$ so that a and b are constants (which recovers the solution (21) with (22)) or

$$a'(p) = -b'(p)\tan p.$$

This equation, together with

$$\left.\begin{aligned} h(p) &= a(p)\cos p + b(p)\sin p, \\ 2(aa'+bb')\left(1-d+\frac{d}{1+aU_1}\right) &= \frac{dU_1(a^2+b^2)}{(1+aU_1)^2}, \\ (a^2+b^2)\,(1-d+d/(1+aU_1)] &= 1, \end{aligned}\right\} \qquad (23)$$

and

generates another solution for h. (This second solution is analogous to the singular solution of the nonlinear equation for $h(\theta)$, (17).) It is a straightforward exercise to confirm that, for $d \to 0$, equations (23) recover the asymptotic solution (20) for which $h(\theta) > 0$ for all $0 \leqslant \theta \leqslant \pi$.

Graphical solutions of (23) are presented in figures 1, 2 and 3 for three different values of U_1 and for up to three different boundary-layer thicknesses, d. The solutions are given for $0 \leqslant \theta \leqslant \pi$ (since the symmetric profile is generated for $-\pi \leqslant \theta \leqslant 0$), where $h(0)$ and $-h(\pi)$ are the initial and final values, respectively, of a. To make clearer the distortion of the wavefront, owing to the presence of the shear flow, each figure includes the corresponding circular-ring wave. All the solutions are expressed in a normalized form such that

$$r = \text{constant}/h(\theta) = h(0)/h(\theta).$$

The figures indicate quite plainly how the presence of the shear flow distorts what would otherwise be a circular-ring wave. We see also that these effects are accentuated as the boundary-layer thickness increases (for a given surface speed). The general character of all these curves is the same no matter whether the flow be subcritical ($U_1 < 1$, figure 1), critical ($U_1 = 1$, figure 2) or supercritical ($U_1 > 1$, figure 3). This similarity has arisen because, in all cases, the generalized Burns condition never produces a solution for which there is a critical layer below the surface (if we start from $h(0) > 0$). However, there is a critical-layer solution if we opt for a different condition at $\theta = 0$ (or at another appropriate value of θ), but we shall argue that the resulting solution is physically unacceptable on a number of counts.

To see how other solutions may arise it is convenient to discuss the details for a given value of U_1 and of d; we choose $U_1 = 2$ and $d = 0.1$. (All other solutions, for $U_1 > 0$ and $0 < \mathrm{d} < 1$, exhibit precisely the same characteristics.) We choose to describe the solutions in terms of a; we find that there is no real solution for $h(\theta)$ if

$$-1.106 > a \quad \text{or} \quad -0.555 < a < -0.485 \quad \text{or} \quad a > 1.036;$$

we have recorded these values to three decimal places. The wavefront shown in figure 3, with $d = 0.1$, is generated by the parameter values $1.036 \geqslant a \geqslant -0.485$. A second real solution occurs for $-0.555 \geqslant a \geqslant -1.106$, and in this case a critical layer is present below the surface for all θ. Indeed, it is easy to see that with $U_1 = 2$ a critical layer will exist for $a < -0.5$. Consequently this alternative solution maps from $\theta = \pi$ to $\theta = \pi$ (minimum $\theta \approx 2.566$) and it has a critical layer everywhere; neither of these properties seem tenable for our problem. First, we are seeking a solution which describes an outward-propagating ring wave, and so we require $h(0) > 0$ (since $c = U_1$); these solutions with critical layers are defined only for $\pi \geqslant \theta \geqslant \theta_0 > 0$ (where $\theta_0 \approx 2.6$ if $U_1 = 2, d = 0.1$). Secondly, there is only one solution with $h(0) > 0$ and this

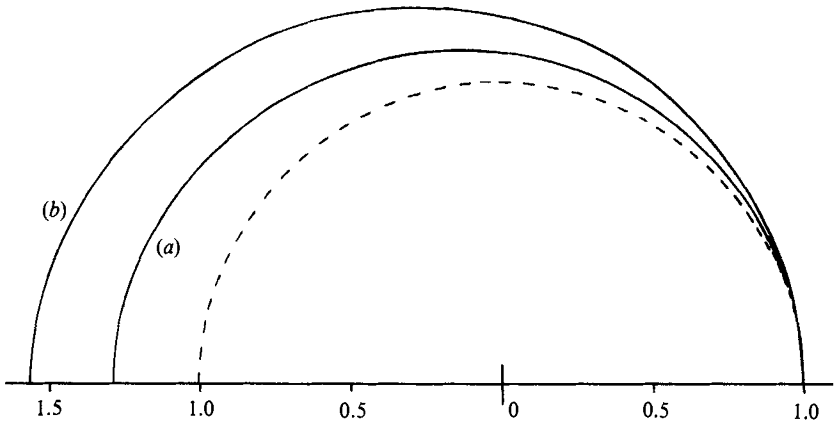

FIGURE 1. Shape of the wavefront ($0 \leqslant \theta \leqslant \pi$) for $U_1 = 0.5$ with (*a*) $d = 0.5$; (*b*) $d = 0.9$. The broken curve is the corresponding circular-ring wave.

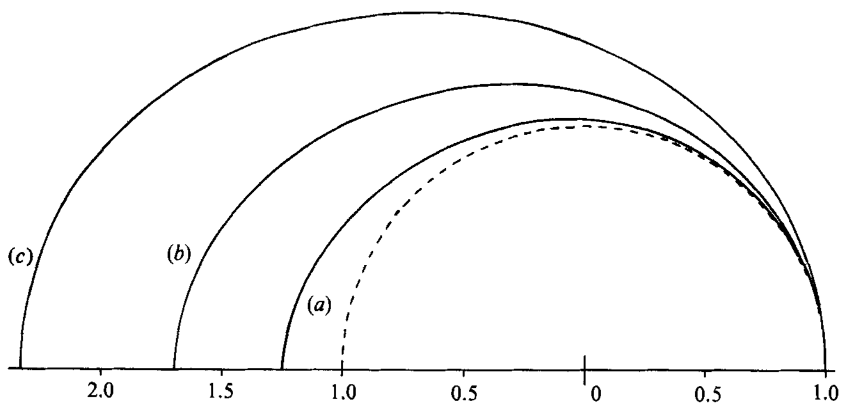

FIGURE 2. Shape of the wavefront ($0 \leqslant \theta \leqslant \pi$) for $U_1 = 1$ with (*a*) $d = 0.1$; (*b*) $d = 0.5$; (*c*) $d = 0.9$. The broken curve is the corresponding circular-ring wave.

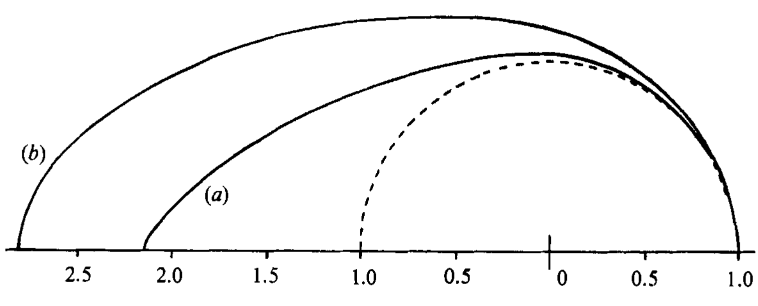

FIGURE 3. Shape of the wavefront ($0 \leqslant \theta \leqslant \pi$) for $U_1 = 2$ with (*a*) $d = 0.1$; (*b*) $d = 0.5$. The broken curve is the corresponding circular-ring wave.

solution does not exhibit a critical layer near $\theta = 0$: any solution of interest here must therefore have no critical layer at least for some θ. The two real solutions are represented in figure 4, where $h(\theta)$ itself is drawn. Of course, the two real branches of the solution are not connected: either there is, or there is not, a critical layer beneath the wavefront for all θ. (It is possible to construct a solution which makes

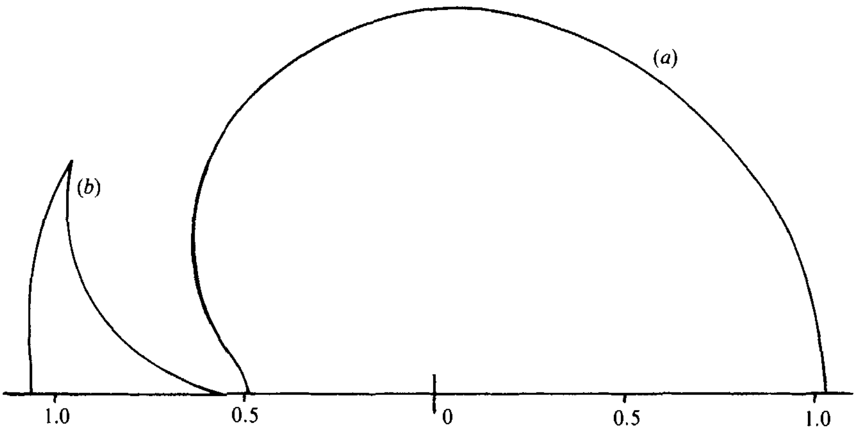

FIGURE 4. The real branches of $h(\theta)$, drawn in polar form, for $U_1 = 2$ and $d = 0.1$. Parameter values (a) $1.036 \geqslant a \geqslant -0.485$; (b) $-0.555 \geqslant a \geqslant -1.106$.

use of one (or more) non-singular solutions (i.e. from (21) and (22)) to connect the two branches, but the resulting solution will exhibit discontinuities both in $h'(\theta)$ and in the height of the critical level.)

5. Nonlinear wave propagation

We now turn our attention to the problem of finding the shape and properties of the surface wave itself, as expressed in far-field variables. We anticipate that the dominant behaviour of η (i.e. η_0), for $\epsilon \to 0$, will exhibit nonlinear and dispersive effects in the form of a KdV-like equation. Furthermore, by virtue of the scaling used to define the far-field region, a contribution from the geometrical decay of the ring wave will also occur at this order. In this section we shall outline the derivation of this equation. The procedure is a very familiar one: it involves finding the equations valid at $O(\epsilon)$ which eventually lead to, when the surface boundary conditions are employed, the elimination of η_1. The remaining equation involves the single unknown function $\eta_0(\xi, R, \theta)$.

From (6)–(11) we find that the $O(\epsilon)$ terms yield

$$\mathrm{D}_1 u_1 + \mathrm{D}_2 u_0 + (U-c)\frac{hv_0}{R}\sin\theta + U'w_1\cos\theta + \mathrm{D}_{30} u_0 + h(p_{1_\xi} + p_{0_R}) = 0,$$

$$\mathrm{D}_1 v_1 + \mathrm{D}_2 v_0 - (U-c)\frac{hu_0}{R}\sin\theta - U'w_1\sin\theta + \mathrm{D}_{30} v_0 + h'(p_{1_\xi} + p_{0_R}) + \frac{h}{R}p_{0_\theta} = 0,$$

$$p_{1_\xi} + \mathrm{D}_1 w_0 = 0,$$

$$hu_{1_\xi} + h'v_{1_\xi} + w_{1_\xi} + hu_{0_R} + \frac{hu_0}{R} + h'v_{0_R} + \frac{h}{R}v_{0_\theta} = 0,$$

with
$$w_1 = 0 \quad \text{on} \quad z = 0,$$

and
$$\left.\begin{aligned} \eta_0 p_{0_z} + p_1 &= \eta_1, \\ h_0 w_{0_z} + w_1 &= \mathrm{D}_1\eta_1 + \mathrm{D}_2\eta_0 + U'(h\cos\theta - h'\sin\theta)\eta_0\eta_{0_\xi} + (\eta u_0 + \eta' v_0)\eta_{0_\xi} \end{aligned}\right\} \quad \text{on} \quad z = 1.$$

Here we have used the same notation as earlier, and in addition D_{30} denotes the operator D_3 expressed in terms of u_0, v_0 and w_0. It then follows directly that, for example,

$$p_1 = \eta_1 + (h^2 + h'^2)\eta_{0_{\xi\xi}} \int_z^1 F^2\left(\int_0^{z1} F^{-2}\,dz_2\right)dz_1,$$

where $F = F(z,\theta)$ is the function defined by (13). We now form the combination $(hD_1 u_1 + h'D_1 v_1)$ from the first two equations, and eliminate $(hu_1 + h'v_1)$ by using the equation of mass conservation. The resulting equation for w_1 is solved, in conjunction with the bottom and surface boundary conditions, and p_1 (as given above) is introduced. This procedure leads to a single equation in η_0 since η_1 is eliminated by virtue of the Burns condition, equation (14). The final equation may be written as

$$A\eta_{0_R} + \frac{B}{R}\eta_0 + \frac{C}{R}\eta_{0_\theta} + D\eta_0\eta_{0_\xi} + E\eta_{0_{\xi\xi\xi}} = 0, \tag{24}$$

where $A, \ldots E$ are coefficients which depend on θ; they are defined by

$$A = 2(h^2+h'^2)I_3, \quad D = -3(h^2+h'^2)^2 I_4,$$

$$E = -(h^2+h'^2)^2 \int_0^1 \int_z^1 \int_0^{z_1} \frac{F^2(z_1,\theta)}{F^2(z,\theta)F^2(z_2,\theta)}\,dz_2\,dz_1\,dz,$$

$$C = \frac{2h}{h'\sin\theta - h\cos\theta}[h(h\sin\theta + h'\cos\theta)I_2 + (h^2+h'^2)I_3\sin\theta],$$

and

$$B = \frac{h(h+h'')(h^2+h'^2)}{(h\cos\theta - h'\sin\theta)^2}(I_2 + 2I_3 + I_4)(1 - 4\sin\theta)\sin\theta + \frac{h(h+h'')}{h'\sin\theta - h\cos\theta}[(h\cos\theta + 3h'\sin\theta)I_2 + 4h'I_3\sin\theta],$$

where

$$I_n = \int_0^1 \frac{dz}{[F(z,\theta)]^n},$$

and we have chosen $c = U(1)$. The generalized Burns condition, expressed in this notation, becomes

$$(h^2+h'^2)I_2 = 1.$$

If a critical layer is present then each I_n is evaluated as the finite part of the integral.

The new KdV, (24), which we have presented here contains a novel feature: the dependence on the angle θ. This appears both by virtue of the θ-dependence of the coefficients and the inclusion of the term in η_{0_θ}. It is now a simple exercise to compare this equation with similar KdV equations derived for simpler geometries. First, we observe that the coefficients A, D and E are (apart from the factors $(h^2+h'^2)^n$, $n = 1, 2$) precisely those obtained for the KdV equation which describes the one-dimensional flow of waves over a shear flow; see Freeman & Johnson (1970). Since there is no θ-dependence in that case, and the choice of frame is not relevant, $F(z,\theta)$ is replaced by $F(z) = U(z) - c$. Secondly, and of more significance, is the comparison of our new equation with the concentric (or cylindrical) KdV equation for water waves. To see the connection we choose $h = 1$ (so that we have circular ring-waves) and take $U(z) = \text{constant} = U(1)$; then $F(z,\theta) = -1$ which gives

$$A = -2, \quad B = -1, \quad C = 0, \quad D = -3, \quad E = -\tfrac{1}{3},$$

and so (24) becomes

$$2\eta_{0_R}+\frac{\eta_0}{R}+3\eta_0\,\eta_{0_\xi}+\tfrac{1}{3}\eta_{0_{\xi\xi\xi}}=0. \tag{25}$$

This is the concentric KdV equation as given by Johnson (1980) where the variables used there correspond with the far-field variables used here.

A number of variants of the KdV equation can be solved completely, for the initial-valve problem, by employing the Inverse Scattering Transform (IST) method. This approach is certainly successful, for example, when applied to the classical KdV eouation

$$u_t-6uu_x+u_{xxx}=0,$$

(which has been written in the usual form). The concentric KdV equation, (25), can be solved in a similar fashion; see Calogero & Degasperis (1978); Drazin & Johnson (1989). However it would appear that our new KdV equation, (24), cannot be solved by using these methods. Numerical solutions of (24), and properties of $\eta_0(\xi, R, \theta)$, are left for future study although a few observations are made in the discussion.

6. Discussion

We have presented a theory for the propagation of a ring wave on the surface of a shear flow. In the long-wave approximation a suitable linear problem has been discussed; this has provided a description of how the shape of the wavefront is distorted by the shear flow below the surface. The technique that we have adopted has resulted in a representation of the wavefront which requires the solution, for $h(\theta)$, of a generalized Burns condition. This solution has been determined for a simple, but fairly realistic, choice of the velocity profile, and various ring-wave solutions have been presented.

Although we have not included the effects of viscosity on the propagating wave, we might reasonably suppose that the gross features of the shape of the wavefront will be supported by observations. Clearly, it is of some interest and significance to see to what extent our theory falls short of the reality. At best, it is to be expected that the shape of the wavefront will develop according to some general characteristic(s) of the shear profile, e.g. perhaps an average measure of the boundary-layer thickness. There may be some virtue in undertaking an experimental/observational study in order to see whether the shape of the wavefront can be used to deduce some information about the flow below the surface.

Our main interest, of course, has been in the technical question relating to the role of the various solutions of the generalized Burns condition. (This integral condition, near $\theta = 0$ and $\theta = \pi$, reduces to essentially the classical Burns condition.) The results presented here have attempted to shed a different light on the question of upstream/downstream propagation. That is, which solutions of the Burns condition are relevant when a critical-layer solution is available? We have shown that the only realistic solution (i.e. with $h(0) > 0$) never exhibits a critical layer below the wavefront. Consequently the rearmost portion of the wavefront moves at the (local) Burns speed associated with upstream propagation. Furthermore, this solution describes a wavefront which is continuous and smooth for all θ. There is another solution but this corresponds to the existence of a critical layer for all $\theta \in [\theta_0, \pi]$, $\theta_0 > 0$, and such a solution does not seem viable. We have argued that a realistic solution will not have a critical layer, in particular, near to the furthest downstream region of the wavefront. The possibility of connecting these two branches of the solution for

$h(\theta)$, which we might term the singular solution, was not seriously entertained. The critical-layer solution, although not relevant to our description of the ring wave, might apply to other propagation phenomena such as the class of solutions with critical layers mentioned by Teles da Silva & Peregrine (1988).

One of the advantages of developing a long-wave far-field theory is that it is then a fairly simple exercise to extend the calculation. So, for example, we are able to find the equation which describes the dominant behaviour of the surface wave; this equation is essentially a balance between nonlinearity, dispersion and geometrical decay. With $\eta \sim \eta_0(\xi, R, \theta)$, as $\epsilon \rightarrow 0$, then we have shown that

$$A\eta_{0_R} + \frac{B}{R}\eta_0 + \frac{C}{R}\eta_{0_\theta} + D\eta_0\eta_{0_\xi} + E\eta_{0_{\xi\xi\xi}} = 0,$$

where the coefficients $A, \dots E$ are functions of θ. The profile and the far field are defined so that ξ is the local characteristic variable, and R is the radial variable (which is therefore analogous to time, for large time). The new and significant feature in this KdV equation is the dependence on θ. Although the detailed analysis of this equation is an important issue, it would seem to be too extensive for us to undertake here. However, a few useful observations can be made which indicate that some measure of headway is possible.

First, let us write

$$\eta_0(\xi, R, \theta) = H_0\left(\xi, R, \int^\theta \frac{A(\theta)}{C(\theta)}\mathrm{d}\theta - \ln R\right),$$

provided $C(\theta) \neq 0$, and then the equation for H_0 is

$$AH_{0_R} + \frac{B}{R}H_0 + DH_0H_{0_\xi} + EH_{0_{\xi\xi\xi}} = 0. \tag{26}$$

Now (26) is nearly the concentric KdV equation, but where the coefficients A, B, D and E are functions of $(\zeta + \ln R)$,

$$\zeta = \int^\theta \frac{A(\theta)}{C(\theta)}\mathrm{d}\theta - \ln R.$$

Note, however, that in (26) the new independent variable ζ appears only as a parameter. A second manoeuvre that would be helpful is to rescale H_0, R and ξ and hence eliminate, as far as possible, the variable coefficients. Unfortunately these coefficients depend on R; if they were functions only of ζ then this would prove successful (although even then (26) could not be transformed into the concentric KdV equation since $A/B \neq 2$ in general). Hence we are forced to conclude that (26), which retains the general characteristics of the concentric KdV equation, is probably expressed in its most convenient form. (Of course, other options are available; if we ignore ζ – a parameter – then the coefficients are functions of R and so there may be a case for writing the solution as $H_0 = \alpha(R)\,G(\beta(R), \xi)$ for suitable $\alpha(R)$, $\beta(R)$.) The properties of the new KdV equation, and other aspects of the problem (such as the inclusion of viscous effects in the linear theory), are left for future study.

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